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About proof-search in intuitionistic natural deduction calculus using partial Skolemization

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Abstract. In this paper, automated proof search in single-conclusion sequential variant of intuitionistic and minimal predicate calculus is considered. In this algorithm, meta-variables and partial Skolemization are used. Theorems of soundness and completeness for the considered algorithm are proved.

1. Introduction

Automated theorem proving is an important area of modern research (see e.g. [1]). Considerable attention is paid to methods of searching for natural classical logical inference based on the use of meta-variables (see e.g. [2]). The use of partial Skolemization has shown high efficiency for such methods (see [3, 4]). In this paper, we extend the approach based on the partial Skolemization to intuitionistic and minimal logics.

The method under consideration is described in the framework of some production system with meta-variables. The deductive problems of this production system are formulated using partial Skolemization. The Skolemization is used for the premises of deductive problems. This Skolem normal form allows us not to lose touch with the desired intuitionistic logical inference. The same approach we consider for minimal logic. We denote the production system with Skolemization for minimal logic by $\mathcal{F}_1$. For such system of intuitionistic logic we use the notation $\mathcal{F}_2$.

Within the framework of the production system, the process of searching for a solution to a problem is associated with the formation of an AND/OR search tree. In essence, a production system is a formulation of a certain algorithm up to a strategy for constructing a search tree. Having fixed a specific strategy for constructing a search tree, we get some implementation of the algorithm formulated in this way. For instance, algorithm [4] uses the strategy of depth first search. Theorem of soundness and completeness for the production systems $\mathcal{F}_i$ is proved.

2. Meta-variable approach

First of all, we describe the meta-variable approach for a wide class of production systems that are coupled with sequential variants of predicate calculus. We are interested in single-conclusion sequent calculus [5], but the theory is valid for any sequent calculus. In particular, we can mention calculus from [6, 7]. A *meta-variable* is a syntactic variable that takes as its values the terms of a first-order language. The *pseudo-terms* of any first-order language Ω are determined

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inductively. Constant symbols of language Ω, variables and meta-variables are pseudo-terms. If f is a n-place function symbol of language Ω and $t_1, \ldots, t_n$ are pseudo-terms of language Ω, then $f(t_1, \ldots, t_n)$ is a pseudo-term. A similar definition of *pseudo-formulas* of language Ω is as follows. Symbols $\perp$ and $\top$ are pseudo-formulas of language Ω. If P is a n-place relation symbol of language Ω and $t_1, \ldots, t_n$ are pseudo-terms of language Ω, then $P(t_1, \ldots, t_n)$ is a pseudo-formula. If A and B are pseudo-formulas, then $(A \,\&\, B)$, $(A \vee B)$, and $(A \supset B)$ are pseudo-formulas. If A is a pseudo-formula and x is a variable, then $(\forall x\, A)$ and $(\exists x\, A)$ are pseudo-formulas.

A *substitution* is a mapping from meta-variables to pseudo-terms. The notation

$$\theta = \begin{pmatrix} X_1 & X_2 & \ldots & X_k \\ t_1 & t_2 & \ldots & t_k \end{pmatrix} \tag{1}$$

refers to a substitution mapping each meta-variable X_i to the corresponding pseudo-term t_i, for $i = 1, \ldots, k$. Applying that substitution to a pseudo-term or pseudo-formula E is written in notation as $E\theta$; it means to (simultaneously) replace every occurrence of each X_i in E by t_i.

We will make no distinction between pseudo-formulas that differ only by renaming bound variables. A substitution σ is a *unificator* for two pseudo-terms or pseudo-formulas E_1 and E_2 if $E_1\sigma = E_2\sigma$. By $\mathbf{MGU}(E_1, E_2)$ we denote the *most general unificator* for E_1 and E_2. It is well-known that the question about the existence of such a unificator is effective decidable [8].

To test whether we can combine substitutions, we give the following definition. Let $\theta_1 = \begin{pmatrix} X_{11}, & \ldots & X_{1n_1} \\ t_{11}, & \ldots & t_{1n_1} \end{pmatrix}, \ldots, \theta_r = \begin{pmatrix} X_{r1}, & \ldots & X_{rn_r} \\ t_{r1}, & \ldots & t_{rn_r} \end{pmatrix}$ be substitutions, $r \geq 2$. Based on $\theta_1, \ldots, \theta_r$ we define two expressions $E_1 = (X_{11}, \ldots, X_{1n_1}, \ldots, X_{r1}, \ldots, X_{rn_r})$ and $E_2 = (t_{11}, \ldots, t_{1n_1}, \ldots, t_{r1}, \ldots, t_{rn_r})$. Then, $\theta_1, \ldots, \theta_r$ are said to be *consistent* iff E_1 and E_2 are unifiable. The substitution $\mathbf{MGU}(E_1, E_2)$ is called a *combination* of $\theta_1, \ldots, \theta_r$.

Let $\mathcal{C}$ be arbitrary sequent calculus with sequents of the kind $\Gamma \to \Delta$, where Γ and Δ are finite sets of formulas. Then the production system $\mathcal{P}_\mathcal{C}$ is defined as follows. First of all, *deductive problems* of $\mathcal{P}_\mathcal{C}$ are sequents with meta-variables of the form $\Gamma \to \Delta$, where Γ and Δ are finite sets of pseudo-formulas. The axioms of the calculus $\mathcal{C}$ are considered as *primitive problems* of $\mathcal{P}_\mathcal{C}$. The *rules for reduction* of tasks into subtasks of $\mathcal{P}_\mathcal{C}$ are obtained as a result of applying the bottom-up inference rules of $\mathcal{C}$, i.e. from conclusion to premises.

Consider the process of obtaining reduction rules in more detail. Reductions are performed by means of inference rules of the form

$$\frac{S_1; \quad S_2; \quad \ldots; \quad S_n}{S}. \tag{2}$$

It means that the problem S reduces to problems $S_1, \ldots, S_n$ (see e.g. [2, 3]). Rules $(\to \forall)$ and $(\exists \to)$ that introduce Skolem constants extend the language. Their application in the process of constructing a search tree $T_0 \subseteq T_1 \subseteq T_2 \ldots$ leads to a hierarchy of languages and corresponding Herbrand universes

$$\begin{array}{rl} & T_0 \subseteq T_1 \subseteq \ldots \subseteq T_n \subseteq T_{n+1} \subseteq \ldots \\ \Omega \subseteq & \Omega_0 \subseteq \Omega_1 \subseteq \ldots \subseteq \Omega_n \subseteq \Omega_{n+1} \subseteq \ldots \\ & H_0 \subseteq H_1 \subseteq \ldots \subseteq H_n \subseteq H_{n+1} \subseteq \ldots \end{array} \tag{3}$$

Rules $(\forall \to)$ and $(\to \exists)$ that introduce new meta-variables extend the set of global meta-variables in a search tree. Each global meta-variable is assigned a *range of values*, which is the Herbrand universe formed at the time the meta-variable was created.

A substitution (1) is called a *correct substitution* for constructing an AND/OR search tree (3) if for every meta-variable X_i such that the range of X_i is H_n holds t_i is a pseudo-term of language Ω_n. A correct substitution (1) is *ground* if $t_i \in H_n$. A deductive problem S with meta-variables $X_1, \ldots, X_k$ is said to be *primitivizable* iff there exists a correct substitution (1) instead of the meta-variables $X_1, \ldots, X_k$ such that the substituted problem $S\theta$ is primitive.

The production system $\mathcal{P}_\mathcal{C}$ is a formulation of a certain algorithm up to a strategy for constructing an AND/OR search tree. Having fixed a specific strategy for constructing a search tree, we get some implementation of the algorithm formulated in this way. The strategy for constructing a search tree consists in the method of selecting a leaf to form, at first, all the substitutions that primitivizes this leaf, at second, all child bundles of nodes corresponding reduction rules. Then, each such substitution "rises" along the branch connecting the leaf to the root, using the combination operation (see e.g. [2, 3]). This forms the set of *admissible substitutions* at the nodes of the search tree: if $\theta_1, \ldots, \theta_n$ are consistent admissible substitutions of nodes $S_1, \ldots, S_n$ respectively, and a problem S is related to problems $S_1, \ldots, S_n$ by a relation (2), then the combination of $\theta_1, \ldots, \theta_n$ is admissible substitution for the node S.

A problem S is called *decidable* or *solvable* if the set of admissible substitutions of the root S of a some search tree is not empty. Theorem of soundness and completeness holds for the production system $\mathcal{P}_\mathcal{C}$ (see e.g. [2, 3]), i.e. the following proposition is true: *a sequent S is derivable in a calculus $\mathcal{C}$ iff the problem S is decidable in the production system $\mathcal{P}_\mathcal{C}$.*

We define the production system $\mathcal{P}_i$, where $i = 1$ or 2, that is coupled with single-conclusion sequent calculus $\mathcal{S}_i$ from [5]. Note that the article [5] contains an error. In the rules of inference $(\supset\to)$, $(\&\to)$, $(\vee\to)$, $(\forall\to)$ and $(\exists\to)$, the formula G can be not only an atom, a disjunction or $\perp$, but also a formula of the form $\exists x\, C(x)$. The system $\mathcal{P}_i$ is the standard for justifying the system $\mathcal{F}_i$. The soundness and completeness of the system $\mathcal{F}_i$ will be established by comparison with $\mathcal{P}_i$. Note that there is an efficient algorithm that maps the derivation subtree from the search tree in $\mathcal{P}_i$ to sequential derivation in $\mathcal{S}_i$. And also there is an efficient algorithm that maps the single-conclusion sequential derivation to proof in natural deduction calculus.

A *deductive problem* of $\mathcal{P}_i$ is a pair (Γ, A), where Γ is a finite set of pseudo-formulas and A is a pseudo-formula. A deductive problem is *primitive* if it is of the kind $(\Gamma, \top)$, or it is of the kind $(G\,\Gamma, G)$, where G is an atomic pseudo-formula or $\perp$.

Now we formulate the *rules of decomposition* for our production systems $\mathcal{P}_i$. Next to the line we indicate the symbolic designation of the rule. The production system $\mathcal{P}_1$ includes reduction rules $(\supset\to)$, $(\to\supset)$, $(\&\to)$, $(\to\&)$, $(\vee\to)$, $(\to\vee)$, $(\exists\to)$, $(\to\exists)$, $(\forall\to)$, and $(\to\forall)$. The production system $\mathcal{P}_2$ is obtained from $\mathcal{P}_1$ by addition of the rule $(\perp_\mathbf{i})$. In formulation of rules below, pseudo-formula G is an atomic, is of the kind $C \vee D$, is of the kind $\exists x\, C$, or $\perp$.

- Reduction rules for simplification and splitting:

$$\frac{(A\supset B)\,\Gamma\to A;\quad B\,(A\supset B)\,\Gamma\to G}{(A\supset B)\,\Gamma\to G}\ (\supset\to),\quad \frac{\Gamma\, A\to B}{\Gamma\to A\supset B}\ (\to\supset),\quad \frac{A\,(A\vee B)\,\Gamma\to G;\quad B\,(A\vee B)\,\Gamma\to G}{(A\vee B)\,\Gamma\to G}\ (\vee\to),$$

$$\frac{\Gamma\to A_i}{\Gamma\to A_1\vee A_2}\ (\to\vee_i),\quad \frac{A\,B\,(A\,\&\,B)\,\Gamma\to G}{(A\,\&\,B)\,\Gamma\to G}\ (\&\to),\quad \frac{\Gamma\to A;\quad \Gamma\to B}{\Gamma\to A\,\&\,B}\ (\to\&),\quad \frac{\Gamma\to\perp}{\Gamma\to G}\ (\perp_\mathbf{i}).$$

- Reduction rules that introduce Skolem constants:

$$\frac{A(c)\,(\exists x\,A(x))\,\Gamma\ \to\ G}{(\exists x\,A(x))\,\Gamma\ \to\ G}\ (\exists\to),\qquad \frac{\Gamma\ \to\ A(c)}{\Gamma\ \to\ \forall x\,A(x)}\ (\to\forall),$$

  where c is a new Skolem constant, i.e. c is a free variable that is used instead of constant.

- Reduction rules that introduce meta-variables:

$$\frac{A(X)\,(\forall x\,A(x))\,\Gamma\ \to\ G}{(\forall x\,A(x))\,\Gamma\ \to\ G}\ (\forall\to),\qquad \frac{\Gamma\ \to\ A(X)}{\Gamma\ \to\ \exists x\,A(x)}\ (\to\exists),$$

  where X is a new global meta-variable.

The definition of the production system $\mathcal{P}_i$ is complete. The system $\mathcal{P}_i$ is a special case of $\mathcal{P}_\mathcal{C}$, where $\mathcal{C}$ is $\mathcal{S}_i$. In particular, the soundness and completeness theorem for the system $\mathcal{P}_i$ with respect to calculus $\mathcal{S}_i$ is true.

3. Production system with Skolemization

Positive and *negative occurrences* of a pseudo-formula in a given pseudo-formula are determined inductively. Occurrence of a pseudo-formula into itself is considered as positive. If the pseudo-formula B is positively (negatively) occurs in the pseudo-formula A, then B is negatively (positively) occurs in pseudo-formula $A \supset A'$ and positively (negatively) occurs in pseudo-formulas $A \,\&\, A'$, $A' \,\&\, A$, $\forall x\, A$, $A \vee A'$, $A' \vee A$, $\exists x\, A$, and $A' \supset A$.

Occurrence of a pseudo-formula C into pseudo-formula D is called *strictly positive* if this occurrence is positive and is not part of antecedent of some implication in D. Strictly positive occurrence of existential quantifier $\exists x$ in D is called *blocking* if this quantifier is part of succedent of some implication in D. Occurrence of C into D is called *strictly positive analytic* if this occurrence, at first, is strictly positive; at second, is not part of the scope of some blocking existential quantifier in D; at third, is not part of the scope of some disjunction in D. A pseudo-formula is said to have the *pure variable property* if no variable both occurs free and occurs bound, and every two quantifiers bound different variables in this pseudo-formula.

Let A be a pseudo-formula. Let $\exists x$ be a strictly positive analytic quantifier of existence in A such that $\exists x$ is not blocking. We assume that A has pure variable property. By definition, the *dimension* of the variable x in A is the number of different universal quantifiers such that $\exists x$ is part of the scope of these quantifiers. Let k be the dimension of the variable x in A, and let f be a k-place function symbol such that the symbol f does not occur into A. Consider the list $\forall z_1$, $\ldots$, $\forall z_k$ of all universal quantifiers in A such that $\exists x$ is part of the scope of these quantifiers in A. *The result of application* to a pair $\langle A,\, x\rangle$ of *Skolem method of removing of existential quantifier* we will call the pseudo-formula $A^- \begin{pmatrix} x \\ f(z_1,\, \ldots,\, z_k) \end{pmatrix}$, where A^- means the result of removing from A the occurrence of $\exists x$. We denote this pseudo-formula by $\mathbf{S}(A,\, x)$.

By $\mathbf{Sk}(A)$ we denote *Skolem normal form* of pseudo-formula A which define by induction on number of logical connectives:

- $\mathbf{Sk}(B\&C) = \mathbf{Sk}(B) \,\&\, \mathbf{Sk}(C)$, $\mathbf{Sk}(B \supset C) = B \supset \mathbf{Sk}(C)$, $\mathbf{Sk}(B \vee C) = B \vee C$, $\mathbf{Sk}(\top) = \top$, $\mathbf{Sk}(\bot) = \bot$.
- If A of the form $\exists x\, B$, then $\mathbf{Sk}(A) = \mathbf{Sk}(\mathbf{S}(A,\, x))$.
- If A of the form $\forall z\, B$, then the calculation process of $\mathbf{Sk}(A)$ is divided into two stages. At the first stage, after removing non blocking strictly positive analytic the existence quantifiers and introducing the Skolem functions, we obtain the pseudo-formula

  $$A' = \mathbf{S}(\ldots(\mathbf{S}(\mathbf{S}(A,\, x_1),\, x_2),\ldots),\, x_l).$$

  At the second stage, after removing all strictly positive analytic the quantifiers of universality, we obtain the pseudo-formula A''. Finally, after introducing new local meta-variables $Z_1,\, \ldots,\, Z_m$, we obtain the desired result $\mathbf{Sk}(A) = A'' \begin{pmatrix} z_1 & \ldots & z_m \\ Z_1 & \ldots & Z_m \end{pmatrix}$.

For instance, the first-order formula $\forall x \forall y \exists v\, ((\exists z\, (P(x,\, z) \;\&\; P(z,\, y))) \supset \exists u\, Q(x,\, u,\, v,\, y))$ converts into the Skolem normal form $(\exists z\, (P(X,\, z) \;\&\; P(z,\, Y))) \supset \exists u\, Q(X,\, u,\, f(X,\, Y),\, Y)$, where f is a new Skolem function, X and Y are new local meta-variables.

Let C be a pseudo-formula, and let D be a strictly positive analytic sub-pseudo-formula of C such that D is an atom, D is a disjunction, D is $\bot$, D is $\top$, or D is a pseudo-formula of the form $\exists x\, A$. Let the sub-pseudo-formula D depend on the list of quantifiers

$$\forall z_1 \ldots \forall z_{n_1} \exists v_1 \forall z_{n_1+1} \ldots \forall z_{n_2} \exists v_2 \forall z_{n_2+1} \ldots \forall z_{n_k} \exists v_k \forall z_{n_k+1} \ldots \forall z_{n_k+m}. \tag{4}$$

Assume that in the process of Skolemization, at first, when removing quantifiers $\exists v_1,\, \ldots,\, \exists v_k$, the new Skolem functions $\varphi_1^{n_1},\, \ldots,\, \varphi_k^{n_k}$ are introduced; at second, when removing quantifiers

$\forall z_1, \ldots, \forall z_{n_k+m}$, the local meta-variables $Z_1, \ldots, Z_{n_k+m}$ are introduced. Then by definition the *protocol of Skolemization for C w.r.t. D* is sequence of pseudo-formulas $P_1, \ldots, P_q$ such that $P_1 = C$, $P_q = D\begin{pmatrix} z_1 & \ldots & z_{n_k+m} & v_1 & \ldots & v_k \\ Z_1 & \ldots & Z_{n_k+m} & \varphi_1^{n_1}(Z_1, \ldots, Z_{n_1}) & \ldots & \varphi_k^{n_k}(Z_1, \ldots, Z_{n_k}) \end{pmatrix}$ and the following propositions are true:

- if $P_j = \forall z_i\, B(z_i)$, then $P_{j+1} = B(Z_i)$;
- if $P_j = \exists v_i\, B(v_i)$ and $\exists v_i$ is not blocking, then $P_{j+1} = B(\varphi_i^{n_i}(Z_1, \ldots, Z_{n_i}))$;
- if $P_j = B_1 \supset B_2$, then $P_{j+1} = B_2$;
- if $P_j = B_1 \,\&\, B_2$ and D is included in B_i, then $P_{j+1} = B_i$.

By definition, the *relation of similarity* is the reflexive transitive closure of the relation between pseudo-formulas when one is obtained from the other as a result of replacing a sub-pseudo-formula of the form $A \,\&\, B$ by $B \,\&\, A$.

For every pseudo-formula C and for every strictly positive analytic sub-pseudo-formula D of C there exists a pseudo-formula

$$\begin{gathered} Q_{11}x_{11} \ldots Q_{1r_1}x_{1r_1}(C_1\, \lambda_1\, (\ldots \\ Q_{s1}x_{s1} \ldots Q_{sr_s}x_{sr_s}(C_s\, \lambda_s\, Q_{s+1,1}x_{s+1,1} \ldots Q_{s+1,r_{s+1}}x_{s+1,r_{s+1}}\, D)) \ldots) \end{gathered} \tag{5}$$

such that the pseudo-formula C is similar to (5), the sequence of quantifiers

$$Q_{11}x_{11}\, \ldots\, Q_{sr_s}x_{sr_s}\, Q_{s+1,1}x_{s+1,1}\, \ldots\, Q_{s+1,r_{s+1}}x_{s+1,r_{s+1}} \tag{6}$$

is equal to (4), $\lambda_i \in \{\&, \supset\}$, and the protocol of Skolemization for (5) w.r.t. D is equal to the protocol of Skolemization for C w.r.t. D. We say that (5) is the *normal form of C w.r.t. D*.

Let C be a pseudo-formula, and let D be a strictly positive analytic sub-pseudo-formula of C such that D is an atom, D is a disjunction, D is $\bot$, D is $\top$, or D is a pseudo-formula of the form $\exists x\, A$. The *prenex form of C w.r.t. D* we will call the result of moving of all quantifiers to the beginning of the pseudo-formula C such that D is part of scope of these quantifiers in C.

It's obvious that if (5) is the normal form of C w.r.t. D and C' is the prenex form of C w.r.t. D, then the normal form of C' w.r.t. D is the following pseudo-formula:

$$Q_{11}x_{11}\, \ldots\, Q_{s+1,r_{s+1}}x_{s+1,r_{s+1}}\, (C_1\, \lambda_1\, (C_2\, \lambda_2\, (\ldots (C_s\, \lambda_s\, D)) \ldots). \tag{7}$$

We say that (7) is the *prenex normal form of C w.r.t. D*. We denote this pseudo-formula by $\mathbf{PNF}(C, D)$.

Assume that $F = \mathbf{Sk}(C)$ and (5) is the normal form of C w.r.t. D. Then the Skolem normal form of the pseudo-formula (5) is the following pseudo-formula:

$$F_1\, \lambda_1\, (F_2\, \lambda_2\, (\ldots (F_s\, \lambda_s\, G)) \ldots), \tag{8}$$

where $G = D\zeta$, $F_i = \tilde{C}_i\zeta$, $\zeta = \begin{pmatrix} z_1 & \ldots & z_{n_k+m} & v_1 & \ldots & v_k \\ Z_1 & \ldots & Z_{n_k+m} & \varphi_1^{n_1}(Z_1, \ldots, Z_{n_1}) & \ldots & \varphi_k^{n_k}(Z_1, \ldots, Z_{n_k}) \end{pmatrix}$, $\tilde{C}_i = C_i$ if $\lambda_i =\supset$, and $\tilde{C}_i = \mathbf{Sk}(C_i)$ if $\lambda_i = \&$. We say that (8) is the *normal form of F w.r.t. G*. Denote the pseudo-formula (8) by $\mathbf{NF}(F, G)$. By definition, the *implicant* of the normal form of F w.r.t. G is the following conjunction: $F_{i_1} \,\&\, F_{i_2} \,\&\, \ldots \,\&\, F_{i_n}$, where $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_n}$ is the sequence of all implications among $\lambda_i \in \{\&, \supset\}$.

We define the production system $\mathcal{F}_i$, where $i = 1$ or 2. A *deductive problem* is a pair $(\mathbf{Sk}(\Gamma)\,?\,G)$, where Γ is a finite set of pseudo-formulas and G is a pseudo-formula. Here all pseudo-formulas from Γ and G not include local meta-variables, but probably include global meta-variables. Note that local meta-variables do not have restrictions on the range of values in the search tree. A deductive problem is *primitive* if it is of the kind $(\Gamma \,?\, \top)$, or it is of the kind $((\&_{j=1}^{j=m} A_j)\, \Gamma \,?\, G)$, where G is equal to A_j for some j, and G is an atomic pseudo-formula or $\perp$.

Now we formulate the *rules of decomposition* for our production systems $\mathcal{F}_i$. Next to the line we indicate the symbolic designation of the rule.

1. The wording of reduction rules $(?\&)$, $(?\supset)$, $(?\vee_i)$, $(?\forall)$, $(?\exists)$, $(\perp_{\mathbf{i}})$ corresponds to the wording of the rules $(\to \&)$, $(\to\supset)$, $(\to \vee_i)$, $(\to \forall)$, $(\to \exists)$, $(\perp_{\mathbf{i}})$:

$$\frac{\Gamma \,?\, A_1;\quad \Gamma \,?\, A_2}{\Gamma \,?\, A_1 \,\&\, A_2}, \qquad \frac{\mathbf{Sk}(A)\, \Gamma \,?\, B}{\Gamma \,?\, A \supset B}, \qquad \frac{\Gamma \,?\, A_i}{\Gamma \,?\, A_1 \vee A_2}, \qquad \frac{\Gamma \,?\, A(c)}{\Gamma \,?\, \forall x\, A(x)}, \qquad \frac{\Gamma \,?\, A(X)}{\Gamma \,?\, \exists x\, A(x)}, \qquad \frac{\Gamma \,?\, \perp}{\Gamma \,?\, G}.$$

2. Let G be a pseudo-formula such that G is an atomic, or G is a disjunction, or G is of the form $\exists x\, B$, or G is equal to $\perp$. Let $D_1 \vee D_2$ be a strictly positive analytic sub-pseudo-formula of a pseudo-formula A. Let the normal form of A w.r.t. $D_1 \vee D_2$ of the kind $A_1\, \lambda_1\, (A_2\, \lambda_2\, (\ldots (A_n\, \lambda_n\, (D_1 \vee D_2)) \ldots)$, where $\lambda_i \in \{\&, \supset\}$. Let $A_{i_1} \,\&\, A_{i_2} \,\&\, \ldots \,\&\, A_{i_k}$ be the implicant of the normal form. Let η be a substitution such that η replaces all the local meta-variables in the pseudo-formula A with new global meta-variables. We denote $A_{i+1}\, \lambda_{i+1}\, (A_{i+2}\, \lambda_{i+2}\, (\ldots (A_n\, \lambda_n\, (D_1 \vee D_2)) \ldots)$ by B_i, where $i = 1, \ldots, n$. By Δ and Δ_j we denote $\Delta_1 = \{\mathbf{NF}(A,\, D_1 \vee D_2),\, A_1,\, B_1,\, A_2,\, B_2,\, \ldots,\, A_{i_1-1},\, B_{i_1-1}\}$, $\Delta_j = \Delta_{j-1} \cup \{B_{i_{(j-1)}},\, \ldots,\, A_{i_j-1},\, B_{i_j-1}\}$, $\Delta = \Delta_k \cup \{B_{i_k},\, \ldots,\, A_n,\, B_n\}$, $B_n = D_1 \vee D_2$, where $j = 1, \ldots, k$. We assume that $(\Delta_1\eta\, A\Gamma \,?\, A_{i_1}\eta)$, $\ldots$, $(\Delta_k\eta\, A\Gamma \,?\, A_{i_k}\eta)$, $(\mathbf{Sk}(D_1\eta)\, (\Delta\eta)\, A\Gamma \,?\, G)$, $(\mathbf{Sk}(D_2\eta)\, (\Delta\eta)\, A\Gamma \,?\, G)$ are solvable problems and $\theta_1, \ldots, \theta_k, \theta_{k+1}, \theta_{k+2}$ are consistent admissible substitutions of these nodes in search tree respectively. Then $(A\, \Gamma \,?\, G)$ is decidable problem too and the combination of $\theta_1, \ldots, \theta_k, \theta_{k+1}, \theta_{k+2}$ is admissible substitution for this node:

$$\frac{\Delta_1\eta\, A\Gamma \,?\, A_{i_1}\eta;\ \ \ldots;\ \ \Delta_k\eta\, A\Gamma \,?\, A_{i_k}\eta;\ \ \mathbf{Sk}(D_1\eta)\, (\Delta\eta)\, A\Gamma \,?\, G;\ \ \mathbf{Sk}(D_2\eta)\, (\Delta\eta)\, A\Gamma \,?\, G}{A\, \Gamma \,?\, G}\ (\vee?).$$

3. Let G be a pseudo-formula such that G is an atomic, or G is a disjunction, or G is of the form $\exists x\, B$, or G is equal to $\perp$. Let $\exists y\, D(y)$ be a strictly positive analytic sub-pseudo-formula of a pseudo-formula A. Let the normal form of A w.r.t. $\exists y\, D(y)$ of the kind $A_1\, \lambda_1\, (A_2\, \lambda_2\, (\ldots (A_n\, \lambda_n\, (\exists y\, D(y))) \ldots)$, where $\lambda_i \in \{\&, \supset\}$. Let $A_{i_1} \,\&\, A_{i_2} \,\&\, \ldots \,\&\, A_{i_k}$ be the implicant of the normal form. Let η be a substitution such that η replaces all the local meta-variables in the pseudo-formula A with new global meta-variables. We denote $A_{i+1}\, \lambda_{i+1}\, (A_{i+2}\, \lambda_{i+2}\, (\ldots (A_n\, \lambda_n\, (\exists y\, D(y))) \ldots)$ by B_i, where $i = 1, \ldots, n$. By Δ and Δ_j we denote $\Delta_1 = \{\mathbf{NF}(A,\, \exists y\, D(y)),\, A_1,\, B_1,\, \ldots,\, A_{i_1-1},\, B_{i_1-1}\}$, $\Delta_j = \Delta_{j-1} \cup \{B_{i_{(j-1)}},\, \ldots,\, A_{i_j-1},\, B_{i_j-1}\}$, $\Delta = \Delta_k \cup \{B_{i_k},\, \ldots,\, A_n,\, B_n\}$, $B_n = \exists y\, D(y)$, where $j = 1, \ldots, k$. Let c be a new free variable in search tree. We assume that $(\Delta_1\eta\, A\Gamma \,?\, A_{i_1}\eta)$, $\ldots$, $(\Delta_k\eta\, A\Gamma \,?\, A_{i_k}\eta)$, $(\mathbf{Sk}(D(c)\eta)\, (\Delta\eta)\, A\Gamma \,?\, G)$ are solvable problems and $\theta_1, \ldots, \theta_k, \theta_{k+1}$ are consistent admissible substitutions of these nodes in search tree respectively. Then $(A\, \Gamma \,?\, G)$ is decidable problem too and the combination of $\theta_1, \ldots, \theta_k, \theta_{k+1}$ is admissible substitution for this node:

$$\frac{\Delta_1\eta\, A\Gamma \,?\, A_{i_1}\eta;\quad \ldots;\quad \Delta_k\eta\, A\Gamma \,?\, A_{i_k}\eta;\quad \mathbf{Sk}(D(c)\eta)\, (\Delta\eta)\, A\Gamma \,?\, G}{A\, \Gamma \,?\, G}\ (\exists?).$$

4. We assume that G and G' are pseudo-formulas such that G and G' are atomic or $\perp$, G and G' are unifiable, and G' is a strictly positive analytic sub-pseudo-formula of a pseudo-formula A. Let η be a substitution such that η replaces all the local meta-variables in the pseudo-formula A with new global meta-variables, $\sigma = \mathbf{MGU}(G,\, G'\eta)$, $\xi = \eta \circ \sigma$ and the normal form of A w.r.t. G' of the kind $A_1\, \lambda_1\, (A_2\, \lambda_2\, (\ldots (A_n\, \lambda_n\, G')) \ldots)$, where $\lambda_i \in \{\&, \supset\}$. Let $A_{i_1} \,\&\, A_{i_2} \,\&\, \ldots \,\&\, A_{i_k}$ be the implicant of the normal form. We denote $A_{i+1}\, \lambda_{i+1}\, (A_{i+2}\, \lambda_{i+2}\, (\ldots (A_n\, \lambda_n\, G')) \ldots)$ by B_i, where $i = 1, \ldots, n$. By Δ_j we denote $\Delta_1 = \{\mathbf{NF}(A,\, G'),\, A_1,\, B_1,\, \ldots,\, A_{i_1-1},\, B_{i_1-1}\}$, $\Delta_j = \Delta_{j-1} \cup \{B_{i_{(j-1)}},\, \ldots,\, A_{i_j-1},\, B_{i_j-1}\}$, where $j = 1, \ldots, k$. We assume that $\Delta_1\xi\, A\Gamma \,?\, A_{i_1}\xi$, $\ldots$,

$\Delta_k\xi\, A\Gamma\,?\,A_{i_k}\xi$ are solvable problems and $\theta_1,\,\ldots,\,\theta_k$ are admissible substitutions of these nodes in search tree respectively such that $\sigma,\,\theta_1,\,\ldots,\,\theta_k$ are consistent substitutions. Then $(A\,\Gamma\,?\,G)$ is decidable problem too and the combination of $\sigma,\,\theta_1,\,\ldots,\,\theta_k$ is admissible substitution for this node:

$$\frac{\Delta_1\xi\, A\Gamma\,?\,A_{i_1}\xi;\quad \Delta_2\xi\, A\Gamma\,?\,A_{i_2}\xi;\quad \ldots;\quad \Delta_k\xi\, A\Gamma\,?\,A_{i_k}\xi}{A\,\Gamma\,?\,G}\ (\supset ?).$$

The definition of the production systems $\mathcal{F}_i$ is complete. The production system $\mathcal{F}_1$ includes reduction rules $(?\&)$, $(?\supset)$, $(?\vee)$, $(?\exists)$, $(?\forall)$, $(\vee ?)$, $(\exists ?)$, and $(\supset ?)$. The production system $\mathcal{F}_2$ is obtained from $\mathcal{F}_1$ by addition of the rule $(\perp_{\mathbf{i}})$.

Theorem of soundness and completeness. *A sequent $\Gamma \to A$ is derivable in $\mathcal{S}_i$ iff the problem $(\mathbf{Sk}(\Gamma)\,?\,A)$ is decidable in $\mathcal{F}_i$.*

The proof of the theorem is easily obtained from two observations 2 and 4, which are consequences of observations 1 and 3, respectively. These observations are formulated below and are proved in the same way as in article [3]. For the production system obtained from $\mathcal{P}_i$ by addition of decomposition rule formulated in observation 1 we use the notation $\mathcal{P}'_i$. We denote the production system obtained from $\mathcal{P}_i$ as the result of generalization to an arbitrary pseudo-formula in a succedent by $\overline{\mathcal{P}}_i$. For the production system obtained from $\mathcal{F}_i$ by addition of decomposition rules formulated in observation 3 we use the notation $\overline{\mathcal{F}}_i$.

Observation 1. *For every pseudo-formula C and for every strictly positive analytic sub-pseudo-formula D of C such that $(\mathbf{PNF}(C,\,D)\,C\,\Gamma,\,A)$ is a decidable problem in $\mathcal{P}_i$ and θ is an admissible substitution for this problem holds the problem $(C\,\Gamma,\,A)$ is decidable in $\mathcal{P}_i$ and θ is an admissible substitution too.*

Observation 2. *If a problem $(\mathbf{Sk}(\Gamma)\,?\,A)$ is decidable in $\mathcal{F}_i$, then the problem $(\Gamma,\,A)$ is decidable in $\mathcal{P}'_i$.*

Observation 3. *In the following six rules, C is an arbitrary pseudo-formula. The first five of them are admissible in $\mathcal{F}_1$. And all six are admissible in $\mathcal{F}_2$:*

$$\frac{\mathbf{Sk}(A)\,\mathbf{Sk}(B)\,\mathbf{Sk}(A\,\&\,B)\,\Gamma\,?\,C}{\mathbf{Sk}(A\,\&\,B)\,\Gamma\,?\,C},\quad \frac{\mathbf{Sk}(A{\supset}B)\,\Gamma\,?\,A;\quad \mathbf{Sk}(B)\,\mathbf{Sk}(A{\supset}B)\,\Gamma\,?\,C}{\mathbf{Sk}(A{\supset}B)\,\Gamma\,?\,C},\quad \frac{\mathbf{Sk}(A(X))\,\mathbf{Sk}(\forall x\,A(x))\,\Gamma\,?\,C}{\mathbf{Sk}(\forall x\,A(x))\,\Gamma\,?\,C},$$

$$\frac{\mathbf{Sk}(A(y))\,\mathbf{Sk}(\exists x\,A(x))\,\Gamma\,?\,C}{\mathbf{Sk}(\exists x\,A(x))\,\Gamma\,?\,C},\quad \frac{\mathbf{Sk}(A)\,\mathbf{Sk}(A{\vee}B)\,\Gamma\,?\,C;\quad \mathbf{Sk}(B)\,\mathbf{Sk}(A{\vee}B)\,\Gamma\,?\,C}{\mathbf{Sk}(A{\vee}B)\,\Gamma\,?\,C},\quad \frac{\Gamma\,?\,\perp}{\Gamma\,?\,C}.$$

Observation 4. *If a problem $(\Gamma,\,A)$ is decidable in $\overline{\mathcal{P}}_i$, then the problem $(\mathbf{Sk}(\Gamma)\,?\,A)$ is decidable in $\overline{\mathcal{F}}_i$.*

4. Conclusion

Earlier the production system of heuristic search for logical classical inference, which uses partial Skolemization, has shown high efficiency (see [3, 4]). In this paper, similar production systems are formulated for intuitionistic and minimal logic. This completes the construction of the theoretical foundation for the effective method of automated reasoning from article [4].

References

[1] 2001 *Handbook of automated reasoning*, ed A Robinson and A Voronkov (Elsevier and MIT Press) p 2122
[2] Konev B Yu and Jebelean T 2005 Solution lifting method for handling meta-variables in the TH∃OREM∀ system *J. Math. Sci. (N. Y.)* **126 3** pp 1182–1194
[3] Okhotnikov O A 2019 About proof-search in classical natural deduction calculus using partial Skolemization (in Russian) *Intelligent systems. Theory and applications* **23 4** pp 39–90
[4] Vtorushin Yu I 2009 O poiske vyvoda v sisteme natural'noj dedukcii logiki predikatov (in Russian) *Intelligent systems* **13 1-4** pp 264–288
[5] Okhotnikov O A 2014 A new sequent calculus for automated proof search *Appl. Math. Sci.* **8 100** pp 4977–4984
[6] Dragalin A G 1988 *Mathematical intuitionism. Introduction to proof theory* (American Mathematical Society)
[7] Takeuti G 1987 *Proof theory* (Dover Publications)
[8] Chang C and Lee R 1973 *Simbolic logic and mechanical theorem proving* (Academic Press)